ERRATA
MATHEMATICS FOR AUSTRALIA 12
Specialist Mathematics
Worked Solutions

First edition - 2017

The following errata were made on 05/May/2020

page 151 EXERCISE 4B Question 10 b, sixth line should read:

\[ 8zz^* = 288 \]

page 156 EXERCISE 4C.1 Question 2, should read:

\[ z = 0 = 0 + 0i \] cannot be written in polar form. The vector representing \( z \) has length zero, and an argument is not defined (no angle can be formed with the positive \( x \)-axis).

page 178 EXERCISE 4E Question 13 c, should read:

\[ z = 2 \cis \left( \frac{\pi}{2} \right) \]

\[ z \times z^2 \times z^3 \times \ldots \times z^k = 2^{ \frac{k(k+1)}{2} } \cis \left[ \frac{k(k+1)\pi}{14} \right] \]

\[ = 2^{ \frac{k(k+1)}{2} } \left[ \cos \left( \frac{k(k+1)\pi}{14} \right) + i \sin \left( \frac{k(k+1)\pi}{14} \right) \right] \]

which is real when \[ \sin \left( \frac{k(k+1)\pi}{14} \right) = 0 \]
\[ \therefore \frac{k(k+1)\pi}{14} = n\pi, \quad n \in \mathbb{Z} \]
\[ \therefore k(k+1) = 14n \]

which has smallest integer solution \( k = 6, \quad n = 3 \)
\[ \therefore \left| z^1 \times z^2 \times z^3 \times \ldots \times z^6 \right| = 2^{15/2} \]
\[ = 2^{21} \]

\[ z \times z^2 \times z^3 \times \ldots \times z^k = 2^{ \frac{k(k+1)}{2} } \left[ \cos \left( \frac{k(k+1)\pi}{14} \right) + i \sin \left( \frac{k(k+1)\pi}{14} \right) \right] \]

which is purely imaginary when \[ \cos \left( \frac{k(k+1)\pi}{14} \right) = 0 \]
\[ \therefore \frac{k(k+1)\pi}{14} = (2n - 1) \frac{\pi}{2}, \quad n \in \mathbb{Z}^+ \]
\[ \therefore \frac{k(k+1)}{14} = \frac{2n - 1}{2} \]
\[ \therefore k(k+1) = 7(2n - 1) \ldots (\ast) \]

But \( k(k+1) \) is even for all \( k \in \mathbb{Z}^+ \) and \( 7(2n - 1) \) is odd for all \( n \in \mathbb{Z}^+ \).

So, \( (\ast) \) is never true.
\[ \therefore z^1 \times z^2 \times z^3 \times \ldots \times z^k \] is never purely imaginary.
4 b ii 1 + w + w^2 + ... + w^{n-1} is a geometric series with \( t_1 = 1 \) and
\[ r = w = \cis\left(\frac{2\pi}{n}\right). \]
\[ \therefore \text{it has sum } S_n = \frac{t_1(r^n - 1)}{r - 1} = \frac{1(w^{n} - 1)}{w - 1} = \frac{\left(\cis\left(\frac{2\pi}{n}\right)\right)^n - 1}{\cis\left(\frac{2\pi}{n}\right) - 1} = \frac{\cis2\pi - 1}{\cis\left(\frac{2\pi}{n}\right) - 1} \{\text{De Moivre}\} = \frac{1 - 1}{\cis\left(\frac{2\pi}{n}\right) - 1} = 0 \]

So, \( 1 + w + w^2 + ... + w^{n-1} = 0 \).

5 Let \( \alpha = r \cis \theta \)
\[ \therefore z^n = r \cis (\theta + k\pi) \text{ where } k \in \mathbb{Z} \]
\[ \therefore z^n = [r \cis(\theta + k\pi)]^{\frac{1}{n}} \]
\[ \therefore z = \frac{1}{n} \cis \left(\frac{\theta + k\pi}{n}\right) \{\text{De Moivre}\} \]
\[ \therefore z = \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \cis \left(\frac{k\pi}{n}\right) \]
\[ \therefore \text{the } n \text{ roots of } z^n = \alpha \text{ are } \frac{1}{n} \cis \left(\frac{\theta}{n}\right), \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \cis \left(\frac{2\pi}{n}\right), \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \cis \left(\frac{4\pi}{n}\right), \ldots, \]
\[ \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \cis \left(\frac{2\pi}{n}(n-1)\right) \{\text{letting } k = 0, 1, 2, \ldots, n - 1\} \]
\[ \therefore \text{the sum of the } n \text{ roots of } z^n = \alpha \text{ is} \]
\[ \frac{1}{n} \cis \left(\frac{\theta}{n}\right) + \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \cis \left(\frac{2\pi}{n}\right) + \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \cis \left(\frac{4\pi}{n}\right) + \ldots + \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \cis \left(\frac{2\pi}{n}(n-1)\right) \]
\[ = \frac{1}{n} \cis \left(\frac{\theta}{n}\right) \left[1 + \cis \left(\frac{2\pi}{n}\right) + \cis \left(\frac{4\pi}{n}\right) + \ldots + \cis \left(\frac{2\pi}{n}(n-1)\right)\right] \]
\[ = 0 \]
these are the nth roots of unity, whose sum = 0 \{using 4\}
The following errata were made on 28/Apr/2020

page 199 REVIEW SET 4B Question 4 c, should read:

\[ |k - ki| = \sqrt{k^2 + (-k)^2} = \sqrt{2k^2} = |k| \sqrt{2} \]

Since \( k < 0 \), \( |k - ki| = -k \sqrt{2} \)

\[ \tan \theta = \frac{k}{k} = 1 \quad \{ k \neq 0 \} \]

\[ \theta = \frac{\pi}{4} \]

\[ \arg(k - ki) = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \]

\[ k - ki = -k \sqrt{2} \cis \left( \frac{3\pi}{4} \right) \]

which is in polar form since \( k < 0 \)

page 203 REVIEW SET 4B Question 17 b, should read:

17 a If \( z = \cis \theta \)
\[ = \cos \theta + i \sin \theta \]
\[ \therefore |z| = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1 \]

b If \( z = \cis \theta \) then \( z^* = \cis(-\theta) \)
\[ = (\cis \theta)^{-1} \quad \{ \text{De Moivre's theorem} \} \]
\[ = z^{-1} = \frac{1}{z} \]

The following errata were made on 28/Apr/2020

page 48 EXERCISE 2C Question 6 b. fourth line should read:

\[ 3z^3 - z^2 + (a + 1)z + a \]
\[ = (3z + 2)(z^2 + bz + c) \quad \text{for some constants} \quad b \quad \text{and} \quad c \]
\[ = 3z^3 + 3bz^2 + 3cz + 2z^2 + 2bz + 2c \]
\[ = 3z^3 + (3b + 2)z^2 + (2b + 3c)z + 2c \]

Equating coefficients gives \[ \begin{cases} 3b + 2 = -1 \quad \ldots \quad (1) \\ 2b + 3c = a + 1 \quad \ldots \quad (2) \\ 2c = a \quad \ldots \quad (3) \end{cases} \]
Using technology, \( x^3 + 2x^2 - 6x - 6 \) has zeros of \(-3.27, -0.860, \) and \(2.13\)
\[\therefore x \approx -3.27, -0.860, \text{ or } 2.13\]

Using technology, \( x^3 + x^2 - 7x - 8 \) has zeros of \(-2.52, -1.18, \) and \(2.70\)
\[\therefore x \approx -2.52, -1.18, \text{ or } 2.70\]

\[\begin{array}{c}
\textbf{2} \quad \textbf{a} \quad 2x = p \\
\therefore x = \frac{1}{2}p \\
&= \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}
\end{array} \quad \begin{array}{c}
\textbf{b} \quad \frac{1}{2}x = p \\
\therefore x = 3p \\
&= \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ -15 \\ 12 \end{pmatrix}
\end{array} \quad \begin{array}{c}
\textbf{c} \quad 4x + p = 0 \\
\therefore 4x = -p \\
\therefore x = \frac{1}{4}p \\
&= \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ -1 \end{pmatrix}
\end{array}\]
3  If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then 

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$b_3$</td>
</tr>
</tbody>
</table>

\[ \therefore \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0 \]

\[ \therefore \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = 0 \]

\[ \therefore \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0 \]

\[ \begin{cases} a_2 b_3 - a_3 b_2 = 0 & \ldots (1) \\ a_1 b_3 - a_3 b_1 = 0 & \ldots (2) \\ a_1 b_2 - a_2 b_1 = 0 & \ldots (3) \end{cases} \]

If $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$, then at least one component of both $\mathbf{a}$ and $\mathbf{b}$ is non-zero. Suppose $a_1, b_1 \neq 0$.

\[ \begin{cases} a_3 = \frac{a_1}{b_1} b_3 \quad \{ (2) \} \\ a_2 = \frac{a_1}{b_1} b_2 \quad \{ (3) \} \\ a_1 = \frac{a_1}{b_1} b_1 \end{cases} \]

\[ \therefore \mathbf{a} = \frac{a_1}{b_1} \mathbf{b}, \quad \frac{a_1}{b_1} \in \mathbb{R}, \quad \frac{a_1}{b_1} \neq 0 \]

We can rearrange (1), (2), and (3) similarly for any pair of non-zero components.

\[ \therefore \mathbf{a} \parallel \mathbf{b} \text{ for any } \mathbf{a}, \mathbf{b} \neq \mathbf{0} \]

If $\mathbf{a} \parallel \mathbf{b}$ then $\mathbf{b} = k \mathbf{a}, \quad k \in \mathbb{R}, \quad k \neq 0$

\[ \therefore \mathbf{a} \times \mathbf{b} = \mathbf{a} \times k \mathbf{a} \]

\[ = k (\mathbf{a} \times \mathbf{a}) \]

\[ = k \times 0 \]

\[ = 0 \]

\[ \therefore \text{if } \mathbf{a} \text{ and } \mathbf{b} \text{ are non-zero vectors then } \mathbf{a} \times \mathbf{b} = 0 \iff \mathbf{a} \text{ is parallel to } \mathbf{b}. \]

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The following erratum was made on 23/Dec/2019

page 154 EXERCISE 4B Question 14 a, second line should read:

14  $a$

\[ z^* = -iz \]

\[ \therefore x - iy = -i(x + iy) \]

\[ \therefore x - iy = -ix + y \]

Equating real and imaginary parts, $x = y$ and $-y = -x$

\[ \therefore y = x \]
The following errata were made on 22/Oct/2019

page 452 EXERCISE 8H Question 5 b ii, should avoid two different uses of the variable name $A$:

\[
\begin{align*}
\text{5 b ii} & \quad \frac{dF}{dt} = kF \left(1 - \frac{F}{95\,000}\right) = kF \left(\frac{95\,000 - F}{95\,000}\right) \\
& \quad \therefore \frac{95\,000}{F(95\,000 - F)} \frac{dF}{dt} = k \\
& \quad \therefore \int \frac{95\,000}{F(95\,000 - F)} dF = \int k \, dt \\
& \quad \therefore \int \left(\frac{1}{F} + \frac{1}{95\,000 - F}\right) dF = \int k \, dt \\
& \quad \therefore \ln |F| - \ln |95\,000 - F| = kt + c \\
& \quad \therefore \ln \left|\frac{F}{95\,000 - F}\right| = kt + c \\
& \quad \therefore \frac{F}{95\,000 - F} = \pm e^{kt+c} \\
& \quad \therefore \frac{95\,000 - F}{F} = Be^{-kt} \quad \left\{ B = \pm \frac{1}{e^c} \right\}
\end{align*}
\]

Now when $t = 0$, $F = 14$

\[
\begin{align*}
& \therefore \frac{95\,000 - 14}{14} = Be^0 \\
& \therefore B = \frac{94\,986}{14}
\end{align*}
\]

In 1900, $t = 55$ and $F = 30\,000$

\[
\begin{align*}
& \therefore 30\,000 = \frac{95\,000}{1 + \frac{94\,986}{14} e^{-55k}} \\
& \therefore 1 + \frac{94\,986}{14} e^{-55k} = \frac{95\,000}{30\,000} \\
& \therefore 1 + \frac{94\,986}{14} e^{-55k} = \frac{19}{6} \\
& \therefore \frac{94\,986}{14} e^{-55k} = \frac{13}{6} \\
& \therefore e^{-55k} = \frac{91}{284\,958} \\
& \therefore -55k = \ln \left(\frac{91}{284\,958}\right) \\
& \therefore k = -\frac{1}{55} \ln \left(\frac{91}{284\,958}\right) = 0.146
\end{align*}
\]

\[
\begin{align*}
& \therefore F \approx \frac{95\,000}{1 + \frac{94\,986}{14} e^{-0.146t}} \\
\end{align*}
\]
8  Let the speed of the particle be \( S(t) \).

\[
S(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}
\]

\[
= \sqrt{(2t-2)^2 + (3t^2-3)^2}
\]

\[
= \sqrt{4t^2 - 8t + 4 + 9t^4 - 18t^2 + 9}
\]

\[
= \sqrt{9t^4 - 14t^2 - 8t + 13}
\]

\[\therefore \quad [S(t)]^2 = 9t^4 - 14t^2 - 8t + 13\]

\[\frac{d}{dt} [S(t)]^2 = 36t^3 - 28t - 8\]

\[= 0 \quad \text{when} \quad t = 1 \quad \{\text{technology, } t > 0\}\]

The maximum and minimum speeds occur either at \( t = 1 \) or at the boundaries.

When \( t = 0 \), \( v = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \)

\[\therefore \quad \text{speed} = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \text{ cm s}^{-1}\]

When \( t = 5 \), \( v = \begin{pmatrix} 8 \\ 72 \end{pmatrix} \)

\[\therefore \quad \text{speed} = \sqrt{8^2 + 72^2} = \sqrt{5248} \text{ cm s}^{-1}\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed</td>
<td>( \sqrt{13} )</td>
<td>0</td>
<td>( \sqrt{5248} )</td>
</tr>
</tbody>
</table>

\[\therefore \quad \text{minimum speed} = 0 \text{ cm s}^{-1},\]

\[\text{maximum speed} = \sqrt{5248} \approx 72.4 \text{ cm s}^{-1}\]

The following erratum was made on 16/Sep/2019

4  \( e^y(2e^2 + 4x + 1) \frac{dy}{dx} = (x + 1)(e^y + 3) \)

\[\therefore \quad \frac{e^y}{e^y + 3} \frac{dy}{dx} = \frac{x + 1}{2x^2 + 4x + 1} \]

\[\therefore \quad \int \frac{e^y}{e^y + 3} \frac{dy}{dx} dx = \int \frac{x + 1}{2x^2 + 4x + 1} dx \]

\[\therefore \quad \int \frac{e^y}{e^y + 3} dy = \frac{1}{4} \int \frac{4x + 4}{2x^2 + 4x + 1} dx \]

\[\therefore \quad \ln |e^y + 3| = \frac{1}{4} \ln |2x^2 + 4x + 1| + c \]

\[\therefore \quad e^y + 3 = A \left| 2x^2 + 4x + 1 \right|^\frac{1}{4} \quad \{A = \pm e^c, \ e^y + 3 > 0 \ \text{for} \ y \in \mathbb{R}\} \]

\[\therefore \quad e^y = A \left| 2x^2 + 4x + 1 \right|^\frac{1}{4} - 3 \]

\[\therefore \quad y = \ln \left[ A \left| 2x^2 + 4x + 1 \right|^\frac{1}{4} - 3 \right] \]

But \( y(0) = 2 \), so \( 2 = \ln(A - 3) \)

\[\therefore \quad e^2 = A - 3 \]

\[\therefore \quad A = e^2 + 3 \]

The particular solution is \( y = \ln \left[ \sqrt[4]{2x^2 + 4x + 1} \left( e^2 + 3 \right) - 3 \right] \).
The following erratum was made on 02/Sep/2019

page 425 EXERCISE 8C Question 4, should read:

4  a If \( y = 2x^2 + c \), then \( \frac{dy}{dx} = 4x \) for any constant \( c \) as required.

    b

    c From a, \( y = 2x^2 + c \) is a general solution to the differential equation.
        The particular solution passes through \( (1, \frac{1}{2}) \), so
        \[ \frac{1}{2} = 2(1)^2 + c \]
        \[ \therefore c = \frac{1}{2} - 2 = -\frac{3}{2} \]
        \[ \therefore \text{the particular solution is } y = 2x^2 - \frac{3}{2} \]

The following erratum was made on 14/Aug/2019

page 415 EXERCISE 8B Question 4, should state correct units:

4  The area of the circular ripple is \( A = \pi r^2 \).
    Differentiating both sides of \( A = \pi r^2 \) with respect to \( t \):
    \[ \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \]
    Since the ripple moves out at a constant speed of 1 m s\(^{-1}\),
    \[ \frac{dr}{dt} = 1 \text{ m s}^{-1} \]
    a When \( r = 2 \), \[ \frac{dA}{dt} = 2\pi \times 2 \times 1 = 4\pi \text{ m}^2 \text{ s}^{-1} \]
        \[ \therefore \text{the circle’s area is increasing at } 4\pi \text{ m}^2 \text{ per second.} \]
    b When \( r = 4 \), \[ \frac{dA}{dt} = 2\pi \times 4 \times 1 = 8\pi \text{ m}^2 \text{ s}^{-1} \]
        \[ \therefore \text{the circle’s area is increasing at } 8\pi \text{ m}^2 \text{ per second.} \]

The following errata were made on 12/Jul/2019

page 89 REVIEW SET 2A Question 11, last line should read:

11  \[ a(z^4 - 6z^3 + 14z^2 - 10z - 7), \quad a \in \mathbb{Q}, \quad a \neq 0 \]

page 178 EXERCISE 4E Question 13, last two lines should read:

13  c  i  \[ \frac{2}{2^{21}} \]
1 c Line 1 has direction vector \( \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix} \) and line 2 has direction vector \( \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \).

As \( \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \) the two lines are parallel. Hence, \( \theta = 0^\circ \).

To see if the lines are coincident, try to find a shared point. When \( t = 0 \), the point on line 1 is \( (0, 3, -1) \).

\[ \therefore \text{ the unique point on line 2 with z-coordinate } -1 \text{ is the point where } 1 + s = -1 \]

\[ \therefore s = -2. \]

This point is \( (-4, -8, -1) \).

Since \( (0, 3, -1) \neq (-4, -8, -1) \), the lines are not coincident.