The following erratum was made on 12/Apr/2018

page 83 EXERCISE K.3 question 6, should read:

6 Consider the alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) where 
\( 0 \leq b_{n+1} \leq b_n \) for all \( n \in \mathbb{Z}^+ \) and \( \lim_{n \to \infty} b_n = 0. \)

a Explain why \( S_2 = b_1 - b_2 \geq 0. \)

b Show that \( S_4 \geq S_2. \) Hence prove that in general, \( S_{2n} \geq S_{2n-2} \) and \( 0 \leq S_2 \leq S_4 \leq \ldots \leq S_{2n} \leq \ldots \)

c Show that \( S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) \ldots - (b_{2n-2} - b_{2n-1}) - b_{2n} \) 
and \( S_{2n} \leq b_1. \)

d Hence prove that \( S_{2n} \) is convergent. Let \( \lim_{n \to \infty} S_{2n} = S. \)

e Show that \( S_{2n+1} = S_{2n} + b_{2n+1}. \)

f Show that if \( \lim_{n \to \infty} b_n = 0 \) then \( \lim_{n \to \infty} S_{2n+1} = S \) and hence \( \lim_{n \to \infty} S_n = S. \)
Similarly, for any negative continuous function which is increasing on \([a, \infty[\),

\[
L = \sum_{i=a}^{\infty} f(i) \leq \int_{a}^{\infty} f(x) \, dx \leq \sum_{i=a}^{\infty} f(i + 1) = U.
\]
The following errata were made on 14/Nov/2016

page 106 SECTION M Example 47, solutions to parts c and d should read:

\[ y = ce^{3x} - 1 \] is a general solution to the differential equation.

The particular solution passes through \((0, 2)\), so
\[ 2 = ce^{3\times0} - 1 \]
\[ c = 3 \]
\[ \therefore \text{the particular solution is } y = 3e^{3x} - 1 \]

\[ \frac{dy}{dx} = 3 + 3y \]
\[ \therefore \text{at the point } (0, 2), \quad \frac{dy}{dx} = 3 + 3 \times 2 = 9 \]
\[ \therefore \text{the gradient of the tangent to the particular solution } y = 3e^{3x} - 1 \text{ at } (0, 2), \text{ is } 9. \]
\[ \therefore \text{the equation of the tangent is } \]
\[ \frac{y - 2}{x - 0} = 9 \]
\[ \therefore y = 9x + 2 \]

page 114 SECTION N Example 53, first line of solution should read:

Since \(OP = PQ\), triangle OPQ is isosceles. Hence [PA] is the perpendicular bisector of [OQ].

The following errata were made on or before 09/Dec/2015

page 171 Worked Solutions EXERCISE K.1, Question 1 d had an invalid first step, replace whole solution with:

Suppose \(\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right)\) converges
\[ \therefore \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right) + \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ converges} \]
This is false \(\therefore \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right)\) diverges.
ERRATA
MATHEMATICS FOR THE INTERNATIONAL STUDENT
MATHEMATICS HL (Option): Calculus

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The following errata were made on or before 18/Aug/2015

page 19 EXERCISE B.3, Question 3 c was bad. Replace with 3 d and add Question 4:

3 Use the Squeeze Theorem to prove that \( \lim_{x \to 0} g(x) = 0 \) for:
a \( g(x) = x^2 \cos \left( \frac{1}{x^2} \right) \)  
b \( g(x) = x \sin \left( \frac{1}{x} \right) \)  
c \( g(x) = \frac{|x|}{1 + x^4} \)

4 a Use the Squeeze Theorem to prove that \( \lim_{x \to 0^+} e^{-\frac{1}{x}} \sin x = 0 \).

b Explain why \( \lim_{x \to 0} e^{-\frac{1}{x}} \sin x \) does not exist.

page 34 SECTION F ROLLE’S THEOREM, Rolle’s Theorem doesn’t require \( f(a) = f(b) = 0 \): ROLLE’S THEOREM

Suppose function \( f : D \to \mathbb{R} \) is continuous on the closed interval \([a, b]\), and differentiable on the open interval \((a, b)\). If \( f(a) = f(b) \), then there exists a value \( c \in ]a, b[ \) such that \( f'(c) = 0 \).

Proof of Rolle’s theorem:

Since \( f \) is continuous on \([a, b]\), it attains both a maximum and minimum value on \([a, b]\). If \( f \) takes values greater than \( f(a) \) on \([a, b]\), let \( f(c) \) be the maximum of these.

Now \( f(b) = f(a) \) and \( f(c) > f(a) \), so \( c \in ]a, b[ \).

Since \( f \) is differentiable at \( c \), \( f \) must have a local maximum at \( x = c \). \( \therefore f'(c) = 0 \).

Similarly, if \( f \) takes values less than \( f(a) \) on \([a, b]\), let the minimum of these be \( f(c) \).

It follows that \( f \) has a local minimum at \( x = c \), and therefore \( f'(c) = 0 \).

Finally, if \( f(x) = f(a) \) for all \( x \in [a, b] \) then clearly \( f'(c) = 0 \) for all \( c \in ]a, b[ \).

By taking \( f(a) = f(b) = 0 \), Rolle’s theorem guarantees that between any two zeros of a differentiable function \( f \) there is at least one point at which the tangent line to the graph \( y = f(x) \) is horizontal.

Rolle’s theorem is a lemma used to prove the Mean Value Theorem.

A lemma is a proven proposition which leads on to a larger result.

page 49 EXERCISE H question 8, should ask for definite intervals within the domain of \( F(x) \):

8 Let \( F(x) = \int_1^x \cos(e^{t^2}) \, dt \), \( x > 1 \). Find exactly:
a \( F'(x) \)  
b \( F'(2) \)  
c \( F' \left( \sqrt{\ln \pi} \right) \)  
d \( F''(x) \)  
e \( F''(2) \)
page 147 Worked Solutions EXERCISE B.1 question 7 c, the limits do not exist:

\[ f(x) = \sin(\frac{1}{x}), \ x \neq 0 \]

\( f(x) \) does not approach any value as \( x \to 0 \) from above or below.

- \( \lim_{x \to 0^-} f(x) \) DNE
- \( \lim_{x \to 0^+} f(x) \) DNE
- \( \lim_{x \to 0} f(x) \) DNE

page 149 Worked Solutions EXERCISE B.3 question 3 d becomes 3 c, and question 4 is added:

4 a As \(-1 \leq \sin x \leq 1\) and \(\frac{1}{e^x} > 0\) for all \(x \in \mathbb{R}\),
\[
\frac{1}{e^x} \leq \frac{\sin x}{e^x} \leq \frac{1}{e^x}
\]
But as \(x \to 0^+\), \(\frac{1}{x} \to \infty\) and \(e^x \to \infty\)
\[
\therefore \quad \lim_{x \to 0^+} \left( \frac{-1}{e^x} \right) = \lim_{x \to 0^+} \left( \frac{1}{e^x} \right) = 0
\]
\[
\therefore \quad \lim_{x \to 0^+} e^{-\frac{1}{x}} \sin x = 0 \quad \text{\{Squeeze Theorem\}}
\]
b As \(-1 \leq \sin x \leq 1\) and \(\frac{1}{e^x} > 0\) for all \(x \in \mathbb{R}\),
\[
\frac{1}{e^x} \leq \frac{\sin x}{e^x} \leq \frac{1}{e^x}
\]
Now \(\lim_{x \to 0^+} e^{-\frac{1}{x}} = 0\) but \(\lim_{x \to 0^-} e^{-\frac{1}{x}}\) is undefined.
\[
\therefore \lim_{x \to 0^-} e^{-\frac{1}{x}}\sin x \text{ is undefined.}
\]
\[
\therefore \lim_{x \to 0} e^{-\frac{1}{x}}\sin x \text{ is undefined.}
\]

page 161 Worked Solutions EXERCISE H question 8, answers need to change according to question alteration:

8 \( F(x) = \int_1^x \cos(e^{t^2}) \, dt, \ x > 1 \)

a \( F'(x) = \cos(e^{x^2}), \ x > 1 \quad \text{\{FTOC\}} \)
b \( F'(2) = \cos(e^4) \)
c \( F'\left(\sqrt{\ln \pi}\right) = \cos\left(\sqrt{\ln \pi}\right) = \cos \frac{\ln \pi}{2} = -1 \)
d \( F''(x) = -\sin(e^{x^2})e^{x^2} 2x = -2xe^{x^2}\sin(e^{x^2}) \)
e \( F'''(2) = -4e^4 \sin(e^4) \)
Thus \( P_n = 2^{n-1} \) is convergent.

But \( P_n = 2^{n-1} \cdot \ln n \) is divergent.

\[ \int f \text{d}x = \int \frac{1}{x \ln x} \, dx = [x \ln x]^{-1}, \quad x \geq 2 \]

\[ f'(x) = -[x \ln x]^{-2} \left( 1 \ln x + x \left( \frac{1}{x} \right) \right) \]

\[ = -\frac{\ln x + 1}{(x \ln x)^2} \]

\[ \Rightarrow b_n = \frac{1}{n \ln n} \]

\[ \Rightarrow b_n \]

\[ \text{is decreasing for all } n \geq 2, \quad n \in \mathbb{Z}^+ \]

Thus \( \sum_{n=2}^{\infty} (-1)^n \ln n \) is convergent.

But \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) is divergent.

\[ \therefore \sum_{n=2}^{\infty} (-1)^n \ln n \]

is conditionally convergent.

---

Consider \( f(x) = \frac{\ln x}{x} \)

Now \( f'(x) = \frac{\frac{1}{x} - \ln x}{x^2} = 1 - \ln x + \frac{1}{x^2} \)

\[ \therefore f'(x) < 0 \text{ for all } x \text{ such that } \ln x > 1, \text{ that is, } x > e \]

Thus \( \left\{ \ln \frac{n}{n} \right\} \) is decreasing for all \( n \geq 3, \quad n \in \mathbb{Z}^+ \)

By l'Hôpital's Rule, \( \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{n \to \infty} \frac{\frac{1}{n}}{1} = 0 \)

\[ \therefore \lim_{n \to \infty} \frac{\ln n}{n} = 0, \quad n \in \mathbb{Z}^+ \]

\[ \therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n} \]

is convergent \{Alternating Series Test\}

But \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) is divergent \{Integral test\}

\[ \therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n} \]

is conditionally convergent.

---

Let \( b_n = \frac{1}{n \ln n} \), then \( \lim_{n \to \infty} b_n = 0 \)

Thus \( \sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n} \) is convergent.

But \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) is divergent \{Integral test\}

\[ \therefore \sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n} \]

is conditionally convergent.

---

Consider \( f(x) = \frac{\ln x}{x} \)

Now \( f'(x) = \frac{\frac{1}{x} - \ln x}{x^2} = 1 - \ln x + \frac{1}{x^2} \)

\[ \therefore f'(x) < 0 \text{ for all } x \text{ such that } \ln x > 1, \text{ that is, } x > e \]

Thus \( \sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n} \) is convergent \{Alternating Series Test\}

But \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) is divergent \{Integral test\}

\[ \therefore \sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n} \]

is conditionally convergent.

---

Consider \( f(x) = e^{-x} \)

\[ \therefore f(0) = 1 \]

\[ f'(x) = -e^{-x}, \quad \therefore f'(0) = -1 \]

\[ f''(x) = e^{-x}, \quad \therefore f''(0) = 1 \]

\[ f'''(x) = -e^{-x}, \quad \therefore f'''(0) = -1 \]

By Maclaurin's theorem,

\[ f(x) = e^{-x} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + R_n(x : 0) \]

\[ \therefore \]